Counting in hypergraphs via regularity inheritance

Ewan Davies

London School of Economics and Political Science

E.S.Davies@lse.ac.uk

August 31, 2015

The regularity method: a summary

- In dense graphs, we apply Szemerédi's regularity lemma to decompose a graph into (ε, d)-regular pairs.
- An accompanying counting lemma allows us to approximate the number of small subgraphs lying across interconnected regular pairs.
- We try this approach in 3-uniform hypergraphs, but need to overcome significant difficulties not present in the graph case.
- The proof of the counting lemma here differs from previous approaches, some simplicity is gained at the expense of using more powerful tools.

Describing the graph case

Definition (Regularity for graphs)

A bipartite graph G on vertex set $V = V_1 \cup V_2$ is (ε, d)-regular if, for all functions $u_i\colon V_i \to [0,1]$, $i = 1,2$ we have

$$\left| \mathbf{E}\left[\big(g(x_1,x_2) - d\big) u_1(x_1) u_2(x_2) \middle| x_i \in V_i \right] \right| \leq \varepsilon.$$

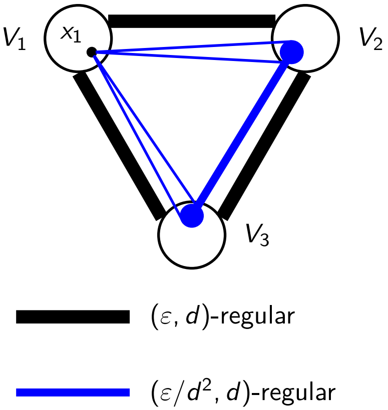
Though it may appear different, this is equivalent to the usual definition of regularity for dense graphs.
If we impose $u_i : V_i \to \{0,1\}$ then the u_i functions indicate subsets of V_i.
The relaxation to $[0,1]$-valued functions makes no difference, by linearity the extrema occur when they are $\{0,1\}$-valued.

Fact (Slicing)

If G as above is (ε, d)-regular, then for $i = 1,2$ and any $U_i \subset V_i$ of size at least $\alpha|V_i|$ the induced subgraph $G[U_1, U_2]$ is $(\varepsilon/\alpha^2, d)$-regular.

Example: counting triangles

A typical $x_1 \in V_1$ of an (ε, d)-regular pair has approximately $d|V_2|$ neighbours in V_2. If $\varepsilon \ll d$ we can count copies of small graphs in collections of regular pairs, embedding vertex-by-vertex.



- By the slicing lemma, a typical $x_1 \in V_1$ has a neighbourhood which is an $(\varepsilon/d^2, d)$-regular pair.
- This can be seen as a form of *regularity interitance*. The neighbourhood of x_1 inherits regularity from the parent system.
- We can estimate the number of triangles containing x_1 as regularity implies bounds on density.

The 3-uniform case: relative quasirandomness

The 3-uniform hypergraph regularity of Frankl and Rödl (1992) decomposes a hypergraph into pieces with the following property.

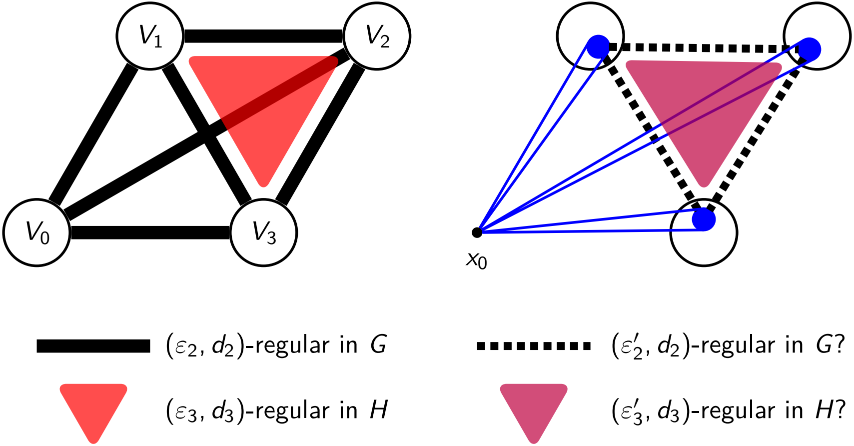
Definition (Regularity for 3-uniform hypergraphs)

Let $V = V_1 \cup V_2 \cup V_3$ be a partition of a vertex set, with each pair of parts (ε_2, d_2)-regular in a graph G. Let H be a 3-uniform hypergraph graph with indicator function $h\colon V \to \{0,1\}$, such that edges of H are triangles in G. We say H is (ε_3, d_3)-regular relative to G if, for all pairs $f = 12, 13, 23$ and functions $u_f\colon V_f \to [0,1]$ with $u_f \le g_f$ pointwise, we have

$$\left| \mathbf{E}\left[(h(x) - d_3) \prod_f u_f(x_f) \middle| x \in V_1 \times V_2 \times V_3 \right] \right| \le \varepsilon_3 d_2^3.$$

A key difficulty is that in general we may only hope to ensure the relation $\varepsilon_2 \ll d_2 \ll \varepsilon_3 \ll d_3$ between parameters. This is weaker than the (δ, r)-regularity introduced later (Frankl–Rödl 2002) in which hyperedges of H must be approximately uniformly distributed over r-tuples of subgraphs of G for some $r \gg 1/d_2$.

The small neighbourhood problem



Approximately a proportion d_2^6 of triples in $V_1 \times V_2 \times V_3$ form triangles of G in the neighbourhood of x_0, much smaller than the error term $\varepsilon_3 d_2^3$ in the definition of regularity of H.

Regularity inheritance

- In order to copy the proof of the counting lemma for dense graphs, we need better understanding of the regularity of neighbourhoods.
- A similar problem occurs in regular graphs that are a dense subgraph of a sparse, very quasirandom graph.
- In this setting, Conlon, Fox and Zhao (2014) proved a form of regularity inheritance via a powerful counting result.
- The unifying concept is that when a regular (hyper)graph is a subgraph of a much more well-behaved quasirandom graph, we may prove regularity inheritance by counting copies of certain subgraphs.
- We apply this approach in 3-uniform hypergraphs.

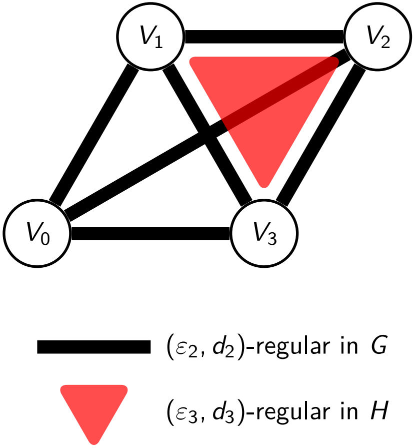
Some notation: for a vertex x, $G(x) = \{y : xy \in G\}$ is the set of vertices which are neighbours of x in a graph G.

Inheritance in 3-uniform hypergraphs

Lemma (D. 2015+)

Consider the graph G and hypergraph H in the adjacent image, and constants $\varepsilon_2 \ll d_2 \ll \varepsilon_3 \ll \varepsilon_3' \ll d_3$.

For all but at most $\varepsilon_3'|V_0|$ vertices $x_0 \in V_0$, the induced 3-graph $H[G(x_0)]$ is (ε_3', d_3)-regular with respect to $G[G(x_0)]$.

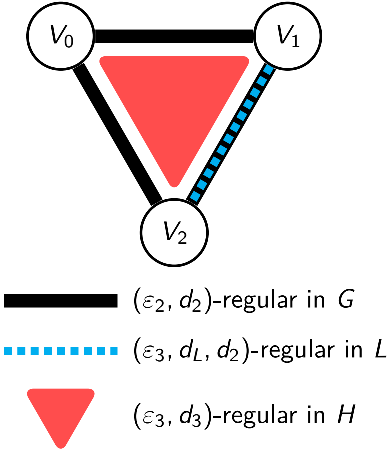


Another inheritance lemma

To prove a 3-uniform hypergraph counting lemma in the spirit of the graph case we need one more lemma which is yet more technical to state.

In essence the lemma states that for a regular subgraph L of $G[V_1, V_2]$, edges of L support approximately the expected number of hyperedges of H.

This implies regularity inheritance for intersections of links in H when L is the link of a vertex.



The 3-uniform counting lemma

The counting lemma we prove with these techniques is a strengthening of that of Frankl and Rödl (2002) as we do not need r-regularity.

Theorem (D. 2015+)

Let J be a set and F be a 3-graph on J. Write ∂F for the union of ∂e over $e \in F$. Let $\{V_j\}_{j \in J}$ be vertex sets each of size at least n. For constants $\frac{1}{n} \ll \varepsilon_2 \ll d_2 \ll \varepsilon_3 \ll \varepsilon_3' \ll d_3$, the following holds. Let G be a graph with indicators $g_f \colon V_f \to \{0,1\}$ which are (ε_2, d_2)-regular for all $f \in \partial F$. Let H be a 3-graph with indicators $h_e \colon V_e \to \{0,1\}$ which are (ε_3, d_3)-regular with respect to G for all $e \in F$.
Then

$$\mathbf{E}\Big[\prod_{e \in F} h_e(x_e) \Big| x \in V_J\Big] = d_3^{|F|} d_2^{|\partial F|} \pm \varepsilon_3' d_2^{|\partial F|}.$$

We proceed vertex-by-vertex in exactly the same manner as for graphs. The details are somewhat technical to express.

Future work: a blow-up lemma

- The blow-up lemma of Komlós, Sárközy and Szemerédi (1997) gives a sufficient condition for the embedding of spanning subgraphs into suitable regular pairs.
- A key part of the proof is a randomised vertex-by-vertex embedding process similar to the proof of the counting lemma.
- Keevash (2011) proved a hypergraph blow-up lemma for embedding spanning subgraphs into regular hypergraphs. His approach differs substantially from that presented here.
- Extending this new method for counting 3-uniform hypergraphs to a new hypergraph blow-up lemma is a work in progress.
- The tools we use for counting small subgraphs and characterising regularity are less well developed in k-uniform hypergraphs for $k > 3$.
- A full treatment of the necessary tools in higher uniformities is a work in progress.